Magic and Mystery of Mathematics

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Mathematics is probably the only subject that can be classified both as art as well as science - former, because it is not constrained by the real world and latter because it is a logical system with precisely defined rules as well as primitives that lead to unambiguous nontrivial theorems. Indian (and of Indian origin) mathematicians have continued to do seminal work till present times, culminating in Manjul Bhargava receiving the Fields Medal last year. In such fabulous times, a non-mathematician ponders about the nature of mathematics, and revisits the question: why are fundamental laws of Nature inherently mathematical?

'Because it is there' of Mathematics

With Manjul Bhargava winning the prestigious Fields Medal, Subhash Khot bagging the Rolf Nevanlinna Prize and Ashoke Sen receiving the Dirac Medal, all in 2014, mathematics has become a source of non-trivial excitement among the young in the land of Srinivasa Ramanujan. Ashoke Sen, an internationally renowned string theorist, is not technically a mathematician. But that is a matter of little consequence since string theory, though a branch of physics, is almost inseparable from proper subsets of advanced pure mathematics. To gauge its importance to mathematics, one may recollect that Edward Witten, *numero uno* of string theory and a stalwart of theoretical physics, was awarded Fields Medal in 1990 for his seminal papers on supersymmetry, Morse and Hodge-de Rham theories, that led to significant progress in number theory and complex analysis[1].

Since antiquity, India has continued to produce many eminent and talented mathematicians like Aryabhatta, Brahmagupta, Bhaskara, Nilakantha,..., Ramanujan, Harish-Chandra, etc. till present times[2]. But why is it that a fraction of gifted people, however minuscule, keep scaling newer heights of an esoteric and abstract subject called mathematics? A George Mallory type answer 'because it is there' notwithstanding, one of the reasons could be the inherent affinity in many for tackling riddles, whodunnits and paradoxes.

Some of us may recall being deeply immersed time to time in resolving conundrums associated with either unexpected-hanging puzzle or Klein bottles or Mobius band from the Late Martin Gardner's feature 'Mathematical Games (which, later, made a metamorphosis to 'Metamagical Themas, an anagram of the original title, in the deft hands of Douglas Hofstadter, after he took charge of the column) that appeared in every issue of the *Scientific American* magazine from mid-50s to 80s. This monthly feature initiated several young students to logical paradoxes like Bertrand Russell's alluring poser involving a fictitious village consisting of barbers who shave only those men who do not shave themselves. Should a barber, belonging to this set, shave himself? As young students, we found ourselves tying into knots fathoming this one! Mathematical theory of knots was of no use in extricating ourselves from this tangled web.

Russell's paradox had serious implications in axiomatic set theory[3]. It paved the way to sharpen concepts in set theory, a field which forms one of the cardinal pillars in all branches of mathematics. It will not be a heresy to state that every branch of mathematics begins with sets and mappings.

Roots

Inception of nascent topics of mathematics like arithmetic and geometry can be attributed to necessity, factoring in human evolution and survival. Archeological evidence suggest that Indus Valley people of $\sim$ 2500 - 1900 BCE used vertical strokes to represent numbers[4]. Early humans not only had to keep track of their possessions through counting but also needed to estimate directions, distances, shapes and sizes, without which hunting, exploration, building cities like that of Harappa and Mohenjo daro, raising humongous Egyptian pyramids, etc, would not have been possible.

Homo sapiens who could assess numbers and sizes, and discern shapes and directions, had an evolutionary advantage for survival during the Darwinian struggle for existence. But as is the wont of human brain, brighter of the lot, dealing with this incipient subject of numbers and shapes, perceived interesting patterns in some of these entities, their interrelations, and posed interesting questions that entailed concepts such as the zero, decimal system (an Indian gift), prime numbers, irrational numbers, the Pythagorean relation between sides and the hypotenuse of right triangles (described also in the *Baudhayana sutras*[5]), and so on.

Playing and tinkering are natural human instincts and, not surprisingly therefore, curious and innovative minds toyed around with patterns found among the arithmetical and geometrical entities to create rich logical systems from which one could establish (i.e. prove) non-trivial results (i.e. theorems) such as the number of prime numbers being

infinite or the Pythagoras theorem, by deploying imaginative and clever tricks on a set of very few, almost self-evident assumptions (i.e. axioms) and by making use of logical rules of operations.

The Midas touch of mathematical minds gave fillip to axiomatic systems, like for example Euclid's geometry, to grow wings as though of their own and to fly out to magical and intangible worlds. Numbers and geometrical concepts got transmuted and generalized to ever increasingly abstract but beautiful creatures, seemingly far removed from concrete reality. New kinds of numbers were imagined. Genie of mathematics was out of the bottle (Klein's?)!

Abstraction and Elegance

Creativity in mathematics displays not only use of abstract concepts and procedures but also elegance and beauty. Puzzling and non-trivial conundrums are often spotted or conjured up, and are eventually settled in ingenious ways. As an illustration, consider the hypothesis H that any irrational number raised to the power of an irrational number is always irrational. Is H true?

Some mathematicians used a brilliant argument to prove that the above hypothesis is false. Take the case when $x = \sqrt{2}$. So, x is manifestly irrational. Construct a new number y,

$$y \equiv x^x \ .$$

Now, clearly if y is rational, the hypothesis H is incorrect, and we are done. On the other hand, if y is irrational (as H will have us believed), define another number,

$$z \equiv y^x \ .$$

Then,

$$z = (x^x)^x = x^{x^2} = (\sqrt{2})^2 = 2 \ ,$$

which is rational.

Thus, using the above counter example(s) one has disproved the hypothesis. But curiously, the preceding argument is completely silent as to whether y is rational or not. All it demonstrates is that if y is irrational then $z = y^x$ is not, when $x = \sqrt{2}$.

Speaking of abstraction, the imaginary number $i \equiv \sqrt{-1}$ (iota) arose as a mathematical device, representing an abstract solution of the quadratic equation $x^2+1 = 0$. It is interesting to note that we can give a geometrical meaning to iota (as the altitude) by interpreting $x^2 + 1 = 0$ to be a Pythagorean relation for an abstract right triangle having a unit base length but a hypotenuse of zero length!

Complex numbers, constructed out of a combination of real numbers and iota, gave birth to a rich structure consisting of elegant theorems involving analytic functions, conformal mappings, analytic continuation, Fourier transforms, Riemann zeta functions, etc. Before the arrival of iota, it was unimaginable that geometrical entities and functions related to logarithm could get linked up, as in the case of the amazing formula due to Euler,

$$e^{i\theta} = \cos\theta + i \sin\theta \ .$$

Complex number was only the beginning. Soon, it was raining quaternions and Grassmann variables!

Till the advent of quantum theory, complex numbers were thought of only as useful tools, having no correspondence with the actual world. After all, in Nature, measurable quantities are always real. But post 1920s, it was realized that the physical world is quantum mechanical in nature, in which the Schrödinger equation,

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi \tag{1}$$

determines the time evolution of a physical system, given the Hamiltonian $\hat{H}$, while the commutator bracket of position and momentum operators,

$$[\hat{x}, \hat{p}] = i\hbar \tag{2}$$

is responsible for Heisenberg's uncertainty principle $\Delta x \Delta p \geq \hbar/2$. Eqs.(1) and (2) emphasize that $i \equiv \sqrt{-1}$ indeed 'exists' in the real world, and it is probably as real as time is[6]. Quantities measured in experiments are still real, as they invariably correspond to eigenvalues of hermitian operators representing the physical observables.

To cite another case, the measure of distance between any two points that relied on Pythagoras theorem in the standard Euclidean geometry got generalized to abstract ones involving metric tensors suitable for curved or warped

spaces described by non-Euclidean geometry. Pioneers of these developments were titans like Gauss, Lobachevsky, Bolyai, Riemann, David Hilbert, Poincare and others. For a comprehensive account of the historical development of the field as well as the monotonic rise of importance of abstract mathematics in physics, readers are referred to N. Mukunda's recent article[7].

A tiny portion of the subject of non-Euclidean geometry came handy when Einstein formulated a relativistic theory of gravity (i.e. general relativity) during 1907-1915. General relativity took geometry of the four dimensional space and time to an exalted level, where it became as much dynamical as the matter itself, whose energy and momentum caused the geometry to be non-Euclidean[8,9].

An oft raised question is whether human beings were genetically programmed to be abstract mathematicians/theoretical physicists. Instead, suppose we ask: were we genetically wired to have been swayed emotionally by sophisticated music? Putting forward a thesis that musical melodies have their roots in the sequence of notes present in bird songs, and that those early humans who were sensitive to and were drawn to simple jingle of a koel's cooing or of other singing birds, had greater chances of survival, finding their mates as well as passing on their genes to offspring (since birds flocked in regions where water and food are abundant), one can possibly explain why music affects us emotionally, and why its primitives are similar to bird songs.

And then with time, simple tunes grew in richness because of the fascinating flexibility of brain which grows more neuronal connections with extra stimuli provided by inputs and outputs from other musically minded people, leading to further creative churnings in music. The ever increasing complex networks both of inter-neuronal highways in the brain as well as of musicians entailed augmentation of a sophisticated body of music. Contemporary music is obviously far more complex and richer in comparison to the brief melody of a bird song. It will not be far fetched to expound a similar theory in the case of mathematics for its ever growing complexity and sophistication. So, what started with an evolutionary advantage, became richer, more abstract and multi-layered over time.

Hilbert, Gödel and 'Hotel Infinity'

David Hilbert, an all time giant of a mathematician, envisaged an axiomatic formulation of mathematics, inspired significantly by the mathematician Moritz Pasch's work, in which all elements of mathematics are divorced entirely from pictures, concrete objects, physical reality, etc., in order that mathematics becomes purely a formal and abstract system so that, starting from a set of axioms, one could prove or disprove all well formed mathematical statements[10].

It is interesting to note that in the celebrated list of 23 'superproblems' which Hilbert had presented in a lecture at Paris in 1900, the 6-th problem was about axiomatic treatment of physics, since its fundamental laws are expressed in the language of mathematics[11]. But his program of a complete axiomatic formulation of mathematics received a jolt when Gödel came up with his incompleteness theorem which, simply stated, implies that in a consistent logical system there will exist well formed logical expressions that can neither be proved nor disproved[10,12].

At this point, one wonders about the effect of Gödel's incompleteness theorem on physical theories which lean heavily on mathematics. Does ambiguity in the physical world enter through a Gödelian back door? Or, is it that whenever an undecidable statement springs up in a physical theory via mathematics, one simply subjects it to an experimental test in order to obtain its truth value? For, natural science has the luxury of experimentation!

Hilbert in 1925 had conjured up a hotel consisting of countable infinity of rooms, bearing room numbers $n = 1, 2, ..., \infty$, in order to illustrate the counter-intuitive nature of a set containing infinite number of elements. He had argued that even if this hotel is fully occupied initially, he could still accommodate M unexpected visitors. All he would do is to shift the existing occupant of room number n to M+n. Then, the new guests could move into the first M vacant rooms. Hilbert went on to claim that even when a countable infinity of visitors arrived suddenly, he could repeat the feat. He would now move the old occupant of room n to 2n, so that an infinity of rooms, bearing odd numbers as room numbers, would lie vacant so that just arrived guests (infinity of them) could move in.

One could possibly extend Hilbert's argument also for an uncountable infinite set (e.g. a set that has the same cardinality as the set of real numbers) trivially using a concrete example. Consider a semi-infinite rod inside a semi-infinite tube, with their ends matching, so that there is no empty space inside the tube. One can create as much space in the tube as one wishes simple by pushing the end of the rod so that it slides forward, while keeping the end of the tube fixed. Apparently George Gamow played a pivotal role in elevating Hilbert's 'hotel infinity' to five star status as the former used it to explain the incessant expansion of an infinite space in the context of big bang model of expanding universe[13].

One may speculate about another possible consequence of 'hotel infinity'. Dirac's theory of electrons and positrons envisages that the vacuum is a 'sea of electrons' filling up all the negative energy states. Suppose there is a physical mechanism (like applying a strong electric field) to shift every electron from its negative energy level to the next lower negative energy state simultaneously so that Pauli's exclusion principle is not violated. Then, Hilbert's argument will create yet another difficulty for the Dirac sea, since moving negative energy electrons to lower levels will entail

+ve energy electrons to fall on to the vacant -ve energy levels releasing gamma photons. It will trigger an unending avalanche of electrons making transitions to more -ve energy states, all heading towards the bottomless pit and generating high energy photons continuously. This makes the physical vacuum unstable, which is an undesirable feature.

Incidentally, it was Hilbert, who proposed an action for general relativistic gravity from which one could also derive, using the principle of least action, Einstein's equations that relate geometry to matter[8]. Another serendipitous Hilbert-physics connection is that according to quantum theory, for every physical system there corresponds an infinite dimensional linear vector space, with complex numbers as the field, which is endowed with an inner product - in short, an example of a Hilbert space!

Physical Sciences and the Unreasonable Effectiveness of Mathematics

Modern mathematics is not a recondite recreation akin to the game of chess (yet another Indian innovation). It is an established fact that all fundamental laws of Nature happen to be expressed in mathematical language. As early as 1959, Eugene Wigner, an eminent theoretical physicist, drove home the point about 'The unreasonable effectiveness of mathematics in the Natural sciences'[14] . Abstract concepts and relations like tensors, affine connection, Hilbert space, operators, matrices, commutators, Lie algebras and groups, Grassmann variables and many more, created by mathematicians for their own sake, turn out to describe fundamental truths concerning the actual universe.

General relativity and the bizarre quantum world of electrons, photons, atoms, etc. bear testimony to the fact that most of the aforementioned abstract concepts 'exist' in the real world. But that is perplexing, as pointed out by Wigner so eloquently, for they come out of mathematicians' brains! However, a crucial fact has been overlooked - only a miniscule portion of the vast body of mathematical creations gets to enjoy the status of being the language of fundamental laws of Nature. The supply of abstract concepts from mathematicians are incredibly larger than the demand from physics. Therefore, should one be surprised that a small subset of mathematical ideas based on logic, beauty, elegance and abstract generalization of concepts, which were rooted to reality once upon a time, gets realized in fundamental physics?

Any idea whose effect or consequences cannot be measured, in principle, is not part of science. It so happens that physical entities are measured quantitatively (i.e. in terms of numbers), and hence, it is natural that their interrelations, including temporal cause and effect connections, better be based on a language that is numeric, precise, logical and unambiguous. Mathematics is such a language and thus, is ideally suited for expressing laws and principles in physical sciences.

Furthermore, since mathematics deals with entities, relations and theorems that are abstract in character, it is naturally amenable to wider applications. For, one can substitute a physical concept in place of an abstract entity, use the established mathematical machinery and arrive at a non-trivial result that is concrete, provided the logical structure of the abstract system is 'isomorphic' to the basic framework of the concrete system. In other words, mathematical modeling of a physical system is often a case of moving from general to particular. To pin point the counter-intuitive nature of any infinite mathematical set that is precisely what Hilbert had done with his example of hotel rooms and guests or what Gamow had attempted when he considered the infinite physical space as an instance of an infinite set. By its very construction, mathematics encompasses such diverse, generalized as well as abstract elements and structures that it has a greater propensity to act as language for physical sciences.

The real world has continued to inspire mathematicians, be it the birth of calculus for finding trajectories of bodies moving in gravitational fields or of distribution theory that ensued from Dirac delta function. It is common wisdom that systematic analysis of gambling outcomes by Fermat, Pascal and Huygens had ushered in the mathematical theory of probability. In a lighter vein, the Mahabharata hero Yudhisthira could have outwitted Shakuni in the game of dice had he brushed up on the theory of chances.

One could speculate whether the chance coincidence of angular sizes of the Sun and Moon being almost same at present (and also during the historical period) causing eclipses to occur, played a significant role in the development of mathematics in the past. After all, mathematicians of great repute exercised their minds to understand and predict eclipses, be it Hipparchus, Aryabhatta or Varahamihira. In other words, it is scientifically relevant to ask whether in the absence of Moon eclipsing the Sun, there would have been enough impetus and motivation to develop trigonometry and other useful computational techniques[15].

The renowned astrophysicist S. Chandrasekhar arrived at many fundamental truths about gravitating systems in astronomy, employing beautiful and elegant mathematical analysis. His predictions concerning upper limit on white dwarf mass, dynamical friction, magnetohydrodynamics instabilities, etc. stood the tests of observational verification, vindicating the power of mathematical rigor[16,17].

Not needing the crutches of experiments, mathematicians rely purely on logical deductions and their thinking prowess, celebrating thereby *Cogito Ergo Sum* of Descartes. Descartes had amalgamated algebra and geometry to

create coordinate geometry. So, on a lighter note, we may juxtapose the utterances of Descartes and Archimedes, to envisage a maxim *'Cogito Ergo Eureka* for the pursuit of mathematics!

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